ZASTOSOWANIA POCHODNEJ FUNKCJI

TWIERDZENIE DE L'HOSPITALA

Niech $f, g : A \to \mathbb{R}$ $(A \subset \mathbb{R})$, gdzie A jest jednym ze zbiorów postaci:

$A = (a, x_0)$, gdzie $-\infty \leq a < x_0 \leq +\infty$
$A = (x_0, b)$, gdzie $-\infty \leq x_0 < b \leq +\infty$
$A = (a, x_0) \cup (x_0, b)$, gdzie $-\infty \leq a < x_0 < b \leq +\infty$.

Zakładamy, że f i g są różniczkowalne na A oraz $g'(x) \neq 0$ dla $x \in A$. Ponadto zakładamy, że zachodzi jeden z następujących warunków:

1) $\lim\limits_{x \to x_0} f(x) = 0 = \lim\limits_{x \to x_0} g(x)$,

2) $\lim\limits_{x \to x_0} |g(x)| = +\infty$.

Wtedy, jeżeli $\lim\limits_{x \to x_0} \frac{f'(x)}{g'(x)} = d \in \overline{\mathbb{R}}$, to także $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = d$.

Zadanie 1 Obliczyć granice funkcji

a) $\lim\limits_{x \to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} = \frac{1 - 1 - 2 \cdot 0}{0 - 0} \overset{H}{=} \left[\frac{0}{0}\right]$

$= \lim\limits_{x \to 0} \frac{[e^x - e^{-x} - 2x]'}{[x - \sin x]'} = \lim\limits_{x \to 0} \frac{e^x - e^{-x} \cdot (-1) - 2}{1 - \cos x} = \frac{1 - 1 \cdot (-1) - 2}{1 - 1} \left[\frac{0}{0}\right] \overset{H}{=}$

$= \lim\limits_{x \to 0} \frac{[e^x + e^{-x} - 2]'}{[1 - \cos x]'} = \lim\limits_{x \to 0} \frac{e^x + e^{-x} \cdot (-1)}{-(-\sin x)} = \frac{1 + 1 \cdot (-1)}{0} \left[\frac{0}{0}\right] \overset{H}{=}$

$= \lim\limits_{x \to 0} \frac{[e^x - e^{-x}]'}{(\sin x)'} = \lim\limits_{x \to 0} \frac{e^x - e^{-x} \cdot (-1)}{\cos x} = \frac{1 - 1 \cdot (-1)}{1} = 2$

b) $\lim\limits_{x \to 0} \frac{x \sin x}{\operatorname{tg} x} = \frac{0 \cdot 0}{0} \left[\frac{0}{0}\right] \overset{H}{=}$

$= \lim\limits_{x \to 0} \frac{[x \sin x]'}{[\operatorname{tg} x]'} = \lim\limits_{x \to 0} \frac{1 \cdot \sin x + x \cdot \cos x}{\frac{1}{\cos^2 x}} = \frac{0 + 0}{\frac{1}{1}} = 0$

c) $\lim\limits_{x \to \frac{\pi}{2}} \frac{\ln(\sin x)}{(\pi - 2x)^2} = \frac{\ln(1)}{0^2} \left[\frac{0}{0}\right] \overset{H}{=}$

$= \lim\limits_{x \to \frac{\pi}{2}} \frac{\frac{1}{\sin x} \cdot \cos x}{2(\pi - 2x) \cdot (-2)} = \frac{\frac{1}{1} \cdot 0}{2 \cdot 0 \cdot (-2)} = \left[\frac{0}{0}\right] H = \lim\limits_{x \to \frac{\pi}{2}} \frac{\left[\frac{\cos x}{\sin x}\right]'}{[-4\pi + 8x]'}$

$= \lim \frac{\frac{-\sin x \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x}}{8} = \lim\limits_{x \to \frac{\pi}{2}} \frac{1}{8} \cdot \frac{-\overbrace{(\sin^2 x + \cos^2 x)}^{=1}}{\sin^2 x} = \frac{1}{8} \cdot \frac{-1}{1^2} = -\frac{1}{8}$

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d) $\lim\limits_{x\to 1^+}\left[\frac{3}{x^3-1}-\frac{4}{x^4-1}\right]=[\infty-\infty]=$

$=\lim\limits_{x\to 1^+}\left[\frac{3(x^4-1)}{(x^3-1)(x^4-1)}-\frac{4(x^3-1)}{(x^3-1)(x^4-1)}\right]=\lim\limits_{x\to 1^+}\frac{3(x^4-1)-4(x^3-1)}{(x^3-1)(x^4-1)}$

$\overset{\left[\frac{0}{0}\right]}{\underset{H}{=}}\lim\limits_{x\to 1^+}\frac{3\cdot 4x^3-4\cdot 3x^2}{3x^2(x^4-1)+(x^3-1)\cdot 4x^3}=\frac{12-12}{0+0}\overset{\left[\frac{0}{0}\right]}{\underset{H}{=}}$

$=\lim\limits_{x\to 1^+}\frac{36x^2-24x}{6x(x^4-1)+3x^2\cdot 4x^3+3x^2\cdot 4x^3+(x^3-1)\cdot 12x^2}=\frac{36-24}{0+12+12+0}=$

$=\frac{12}{24}=\frac{1}{2}$ //

e) $\lim\limits_{x\to 0}\left(\frac{1}{x\cdot\sin x}-\frac{1}{x^2}\right)=[\infty-\infty]=\lim\limits_{x\to 0}\frac{x-\sin x}{x^2\sin x}\overset{\left[\frac{0}{0}\right]}{\underset{H}{=}}$

$=\lim\limits_{x\to 0}\frac{[x-\sin x]'}{[x^2\sin x]'}=\lim\limits_{x\to 0}\frac{1-\cos x}{2x\sin x+x^2\cos x}=\frac{1-1}{0+0}\overset{\left[\frac{0}{0}\right]}{\underset{H}{=}}$

$=\lim\limits_{x\to 0}\frac{[1-\cos x]'}{[2x\sin x+x^2\cos x]'}=\lim\limits_{x\to 0}\frac{-(-\sin x)}{2\sin x+2x\cos x+2x\cos x+x^2\cdot(-\sin x)}$

$=\lim\limits_{x\to 0}\frac{\sin x}{\sin x\cdot(2-x^2)+4x\cos x}=\frac{0}{0+0}\overset{\left[\frac{0}{0}\right]}{\underset{H}{=}}\lim\limits_{x\to 0}\frac{[\sin x]'}{[(2-x^2)\sin x+4x\cos x]'}$

$=\lim\limits_{x\to 0}\frac{\cos x}{-2x\sin x+(2-x^2)\cos x+4\cos x+4x\cdot(-\sin x)}=\frac{1}{0+2+4+0}=\frac{1}{6}$ //

f) $\lim\limits_{x\to 0^+}\sqrt{x}\ln x=[0\cdot(-\infty)]=\lim\limits_{x\to 0^+}\frac{\ln x}{x^{-1/2}}\overset{\left[\frac{-\infty}{\infty}\right]}{\underset{H}{=}}\lim\limits_{x\to 0^+}\frac{[\ln x]'}{[x^{-1/2}]'}$

$=\lim\limits_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}}=\lim\limits_{x\to 0^+}-2\sqrt{x}=0$ //

g) $\lim\limits_{x\to\infty}(\pi-2\operatorname{arctg}x)\cdot\ln x=[0\cdot\infty]=\lim\limits_{x\to\infty}\frac{\ln x}{\frac{1}{\pi-2\operatorname{arctg}x}}\overset{\left[\frac{\infty}{\infty}\right]}{\underset{H}{=}}$

$=\lim\limits_{x\to\infty}\frac{[\ln x]'}{\left[\frac{1}{\pi-2\operatorname{arctg}x}\right]'}=\lim\limits_{x\to\infty}\frac{\frac{1}{x}}{\frac{-1}{(\pi-2\operatorname{arctg}x)^2}\cdot\left(-2\cdot\frac{1}{1+x^2}\right)}=$

$=\lim\limits_{x\to\infty}\frac{(\pi-2\operatorname{arctg}x)^2}{\frac{2x}{1+x^2}}\overset{\left[\frac{0}{0}\right]}{\underset{H}{=}}\lim\limits_{x\to\infty}\frac{\left[(\pi-2\operatorname{arctg}x)^2\right]'}{\left[\frac{2x}{1+x^2}\right]'}=$

$=\lim\limits_{x\to\infty}\frac{2(\pi-2\operatorname{arctg}x)\cdot\left(-2\cdot\frac{1}{1+x^2}\right)}{\frac{2\cdot(1+x^2)-2x\cdot 2x}{(1+x^2)^2}}=\lim\limits_{x\to\infty}2(\pi-2\underbrace{\operatorname{arctg}x}_{\frac{\pi}{2}})\cdot\left(\frac{-2}{\cancel{1+x^2}}\right)\cdot\frac{(1+x^2)^{\cancel{2}}}{\cancel{2}(1-x^2)}$

$\left\{\lim\limits_{x\to\infty}\frac{1+x^2}{1-x^2}=\lim\limits_{x\to\infty}\frac{\cancel{x^2}\left(\frac{1}{x^2}+1\right)}{\cancel{x^2}\left(\frac{1}{x^2}-1\right)}=\frac{1}{-1}=-1\right.$

$=0\cdot(-1)=0$ //

h) $\lim\limits_{x\to 0^+} \left(\ln \frac{1}{x}\right)^x = [\infty^0] = \lim\limits_{x\to 0^+} e^{x\cdot \ln(\ln\frac{1}{x})} = e^{[0\cdot\infty]}$

$= \lim\limits_{x\to 0^+} e^{\frac{\ln(\ln\frac{1}{x})}{\frac{1}{x}}} = e^{\lim\limits_{x\to 0^+} \frac{\ln(\ln\frac{1}{x})}{\frac{1}{x}}} = e^{\boxed{}}$

$\lim\limits_{x\to 0^+} \frac{\ln(\ln\frac{1}{x})}{\frac{1}{x}} \overset{[\frac{\infty}{\infty}]}{\underset{H}{=}} \lim\limits_{x\to 0^+} \frac{[\ln(\ln\frac{1}{x})]'}{[\frac{1}{x}]'} = \lim\limits_{x\to 0^+} \frac{\frac{1}{\ln\frac{1}{x}}\cdot\frac{1}{\frac{1}{x}}\cdot\cancel{\left(\frac{-1}{x^2}\right)}}{\cancel{-\frac{1}{x^2}}}$

$= \lim\limits_{x\to 0^+} \frac{x}{\ln\frac{1}{x}} = \frac{0}{\infty} = \lim\limits_{x\to 0^+} x\cdot\frac{1}{\ln\frac{1}{x}} = 0$

$= e^0 = 1$

i) $\lim\limits_{x\to 0^+} \left(\mathrm{tg}\left(\frac{x}{2}\right)\right)^{\frac{1}{\ln x}} = [0^\infty] = e^{\lim\limits_{x\to 0^+}\frac{1}{\ln x}\cdot\ln\left(\mathrm{tg}\left(\frac{x}{2}\right)\right)} = e^{\boxed{}}$

$\lim\limits_{x\to 0^+} \frac{\ln\left(\mathrm{tg}\left(\frac{x}{2}\right)\right)}{\ln x} \overset{[\frac{-\infty}{-\infty}]}{\underset{H}{=}} \lim\limits_{x\to 0^+} \frac{\left[\ln\left(\mathrm{tg}\left(\frac{x}{2}\right)\right)\right]'}{[\ln x]'} =$

$= \lim\limits_{x\to 0^+} \frac{\frac{1}{\mathrm{tg}(\frac{x}{2})}\cdot\frac{1}{\cos^2(\frac{x}{2})}\cdot\frac{1}{2}}{\frac{1}{x}} = \lim\limits_{x\to 0^+} \frac{x}{2\,\frac{\sin(\frac{x}{2})}{\cancel{\cos(\frac{x}{2})}}\cdot\cos^{\cancel{2}}(\frac{x}{2})} =$

$= \lim\limits_{x\to 0^+} \frac{x}{2\sin(\frac{x}{2})\cos(\frac{x}{2})} = \{\sin(2\alpha) = 2\sin\alpha\cos\alpha\}$

$= \lim\limits_{x\to 0^+} \frac{x}{\sin x} \overset{[\frac{0}{0}]}{\underset{H}{=}} \lim\limits_{x\to 0^+} \frac{[x]'}{[\sin x]'} = \lim\limits_{x\to 0^+} \frac{1}{\cos x} = \frac{1}{1} = 1$

$= e^1 = e$

TWIERDZENIE

Niech $f: I \to \mathbb{R}$ $(I \subset \mathbb{R})$ będzie funkcją różniczkowalną na I.

$[f'(x) > 0$ dla $x \in I] \Rightarrow [f$ jest rosnąca na $I]$

$[f'(x) < 0$ dla $x \in I] \Rightarrow [f$ jest malejąca na $I]$

Jeżeli pochodna zmienia znak przy przechodzeniu przez pewien punkt x_0, to funkcja osiąga w tym punkcie ekstremum lokalne.

Zadanie 2 Wyznaczyć ekstrema lokalne oraz zbadać monotoniczność funkcji

a) $f(x) = -2x^4 + 24x^3 - 92x^2 + 120x + 2$

$D_f = \mathbb{R}$

$f'(x) = 0$

$-8x^3 + 72x^2 - 184x + 120 = 0 \quad / : (-8) \qquad D_{f'} = \mathbb{R}$

$\underbrace{x^3 - 9x^2 + 23x - 15}_{g(x)} = 0$

$g(1) = 1^3 - 9\cdot 1^2 + 23\cdot 1 - 15 = 0 \Rightarrow x_0 = 1$ jest pierwiastkiem $g(x)$.

	1	-9	23	-15
1		1	-8	15
	1	-8	15	=

pierwiastek

reszta

$\Rightarrow g(x) = (x-1)(x^2 - 8x + 15)$

$\Delta = (-8)^2 - 4\cdot 1\cdot 15 = 64 - 60 = 4$

$\sqrt{\Delta} = 2$

$x_1 = \frac{8-2}{2} = 3, \quad x_2 = \frac{8+2}{2} = 5$

$g(x) = (x-1)(x-3)(x-5)$

$f'(x) = -8\cdot g(x)$

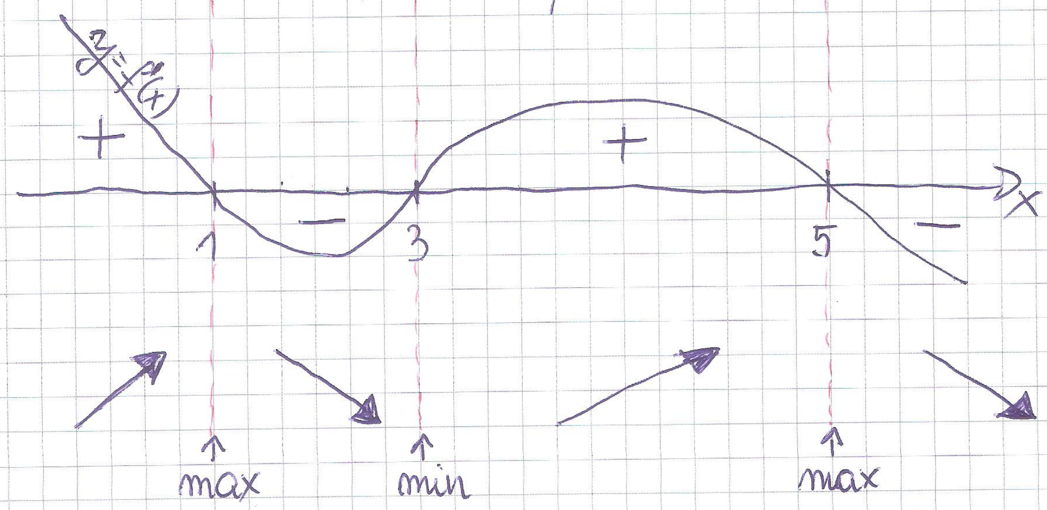
$f'(x) = -8(x-1)(x-3)(x-5)$

$f\nearrow$: $f'(x) > 0$

$-8(x-1)(x-3)(x-5) > 0$

$f\searrow$: $f'(x) < 0$

$-8(x-1)(x-3)(x-5) < 0$



f:

Odp. Funkcja f rośnie dla $x \in (-\infty, 1)$, $x \in (3, 5)$;

f maleje dla $x \in (1, 3)$, $x \in (5, +\infty)$.

Ekstrema lokalne: $f_{max}(1) = 52$, $f_{min}(3) = 20$, $f_{max}(5) = 52$

b) $f(x) = 3x^4 + 16x^3 + 30x^2 + 24x \quad , D_f = \mathbb{R}$

$f'(x) = 12x^3 + 48x^2 + 60x + 24 \quad , D_{f'} = \mathbb{R}$

$f'(x) = 12(x^3 + 4x^2 + 5x + 2)$

	1	4	5	2
-1		-1	-3	-2
	1	3	2	=

$f'(x) = 12(x+1)(x^2+3x+2)$

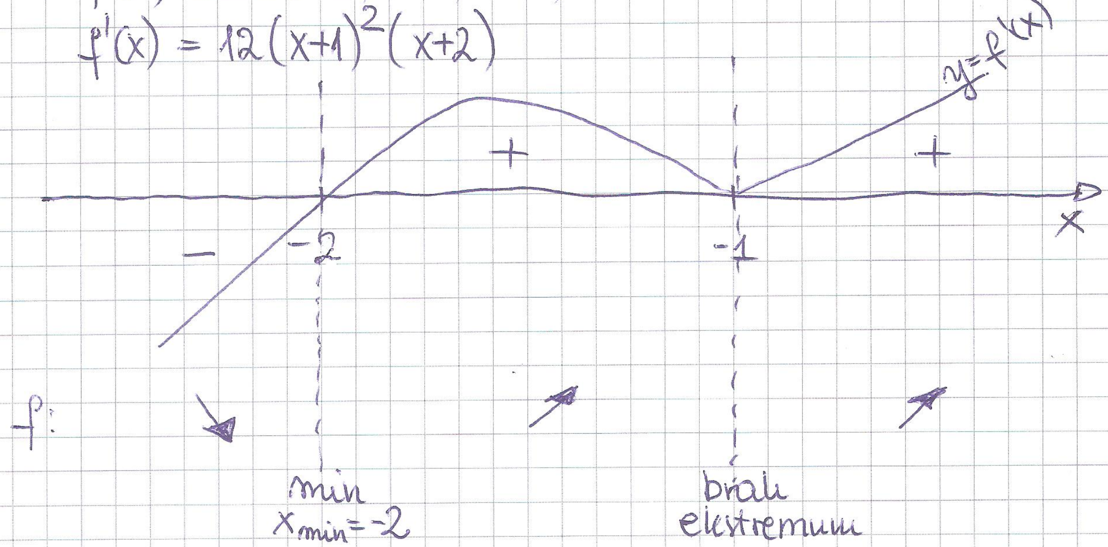
$\Delta = 3^2 - 4\cdot 1\cdot 2 = 9 - 8 = 1$

$x_1 = \frac{-3-1}{2} = -2$

$x_2 = \frac{-3+1}{2} = -1$

$f'(x) = 12(x+1)(x+1)(x+2)$

$f'(x) = 12(x+1)^2(x+2)$



Odp. Funkcja f rośnie dla $x \in (-2,-1)$, $x \in (-1,+\infty)$;
f maleje dla $x \in (-\infty,-2)$.

Ekstrema lokalne jakie osiąga funkcja f to:

$f_{min}(-2) = -8$

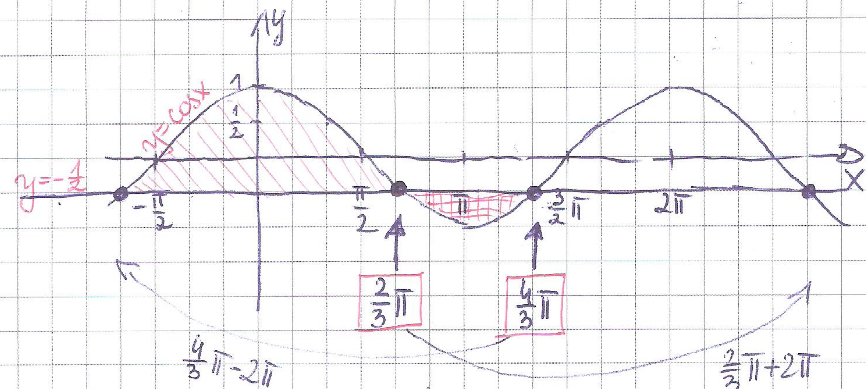
c) $f(x) = x + 2\sin x \quad , \quad D_f = \mathbb{R}$

$f'(x) = 1 + 2\cos x \quad , \quad D_{f'} = \mathbb{R}$

$1 + 2\cos x = 0$

$2\cos x = -1$

$\cos x = -\frac{1}{2}$



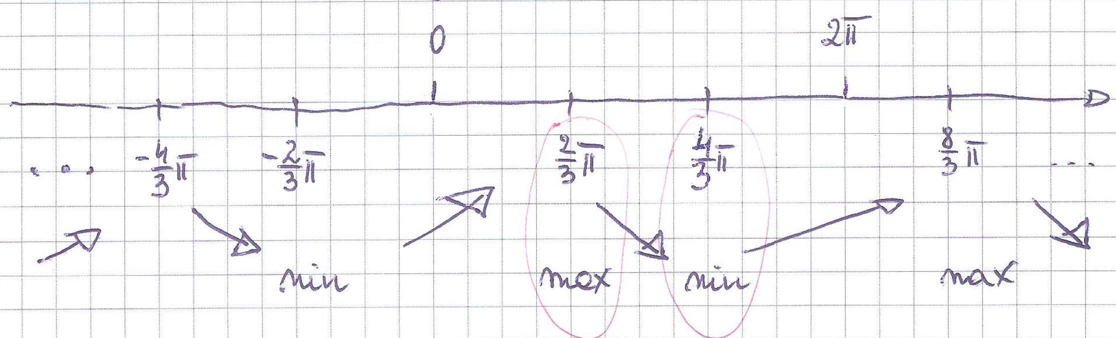
$x = \frac{2}{3}\pi + 2k\pi, \ k \in \mathbb{Z}$ lub $x = \frac{4}{3}\pi + 2k\pi, \ k \in \mathbb{Z}$

$f \nearrow : \ f'(x) > 0$ $\cos x > -\frac{1}{2}$	$f \searrow : \ f'(x) < 0$ $\cos x < -\frac{1}{2}$

$f \nearrow : x \in \left(-\frac{2}{3}\pi + 2k\pi ; \frac{2}{3}\pi + 2k\pi\right), \ k \in \mathbb{Z}$

$f \searrow : x \in \left(\frac{2}{3}\pi + 2k\pi ; \frac{4}{3}\pi + 2k\pi\right), \ k \in \mathbb{Z}$

Rysunek pomocniczy:



$f_{max}\left(\frac{2}{3}\pi + 2k\pi\right) = \left(\frac{2}{3}\pi + 2k\pi\right) + 2\sin\left(\frac{2}{3}\pi + 2k\pi\right) = \left(\frac{2}{3}\pi + 2k\pi\right) \cdot \frac{\sqrt{3}}{2}, \ k \in \mathbb{Z}$

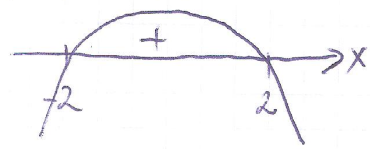
$f_{min}\left(\frac{4}{3}\pi + 2k\pi\right) = \left(\frac{4}{3}\pi + 2k\pi\right) + 2\sin\left(\frac{4}{3}\pi + 2k\pi\right) = \left(\frac{4}{3}\pi + 2k\pi\right) \cdot \left(-\frac{\sqrt{3}}{2}\right), \ k \in \mathbb{Z}$

$\begin{cases} \sin\left(\frac{4}{3}\pi + 2k\pi\right) = \sin\frac{4}{3}\pi = \sin\left(\pi + \frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2} \\ \sin\left(\frac{2}{3}\pi + 2k\pi\right) = \sin\left(\pi - \frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \end{cases}$

a) $f(x) = x\sqrt{4-x^2}$

$D_f: \ 4-x^2 \geq 0$

$(2-x)(2+x) \geq 0$



$D_f = \langle -2, 2 \rangle$

$$f'(x) = 1\cdot\sqrt{4-x^2} + x\cdot\frac{1}{2\sqrt{4-x^2}}\cdot(-2x) = \sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}} =$$

$$= \frac{4-x^2}{\sqrt{4-x^2}} - \frac{x^2}{\sqrt{4-x^2}} = \frac{4-2x^2}{\sqrt{4-x^2}} \qquad , D_{f'} = (-2,2)$$

$$\frac{4-2x^2}{\sqrt{4-x^2}} = 0$$

$4-2x^2=0$

$2-x^2=0$

$(\sqrt{2}-x)(\sqrt{2}+x)=0 \quad \Rightarrow \quad x=\sqrt{2} \ \vee \ x=-\sqrt{2}$

$f\nearrow: \ f'(x)>0$

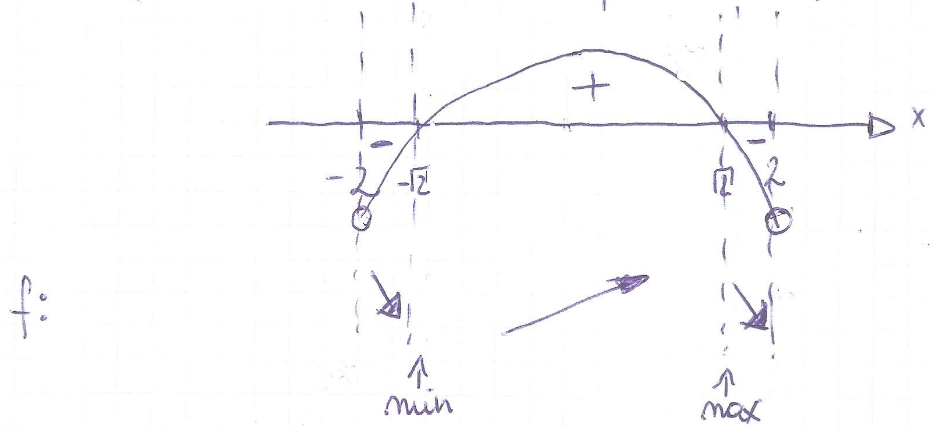
$$\frac{4-2x^2}{\sqrt{4-x^2}} > 0 \quad \Big| \cdot \sqrt{4-x^2} \ (>0)$$

$(\sqrt{2}-x)(\sqrt{2}+x) > 0$

$f\searrow: \ f'(x)<0$

$\vdots$

$(\sqrt{2}-x)(\sqrt{2}+x) < 0$



$x_{min} = -\sqrt{2}, \quad x_{max} = \sqrt{2}$

$f_{min}(-\sqrt{2}) = -2$

$f_{max}(\sqrt{2}) = 2$

e) $f(x)=\dfrac{x^2}{x^2-1}$ $D_f=\mathbb{R}\setminus\{-1,1\}$

$f'(x)=\dfrac{2x\cdot(x^2-1)-x^2\cdot 2x}{(x^2-1)^2}$ $D_{f'}=\mathbb{R}\setminus\{-1,1\}$

$f'(x)=\dfrac{-2x}{(x^2-1)^2}=0$

$-2x=0$

$x=0$

$f\nearrow:\ f'(x)>0$

$\dfrac{-2x}{(x^2-1)^2}>0$

$-2x>0$

$x<0$

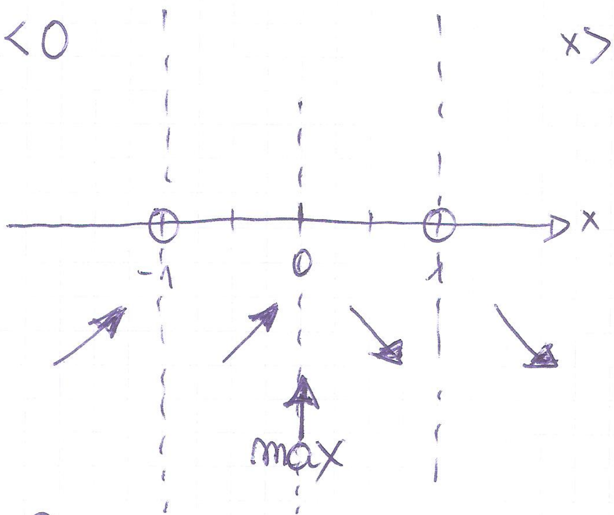
$f\searrow:\ f'(x)<0$

$\dfrac{-2x}{(x^2-1)^2}<0$

$-2x<0$

$x>0$

f:



$f_{max}(0)=\dfrac{0}{0^2-1}=0$

f) $f(x) = \dfrac{e^x}{x+4}$ $\qquad D_f = \mathbb{R} \setminus \{-4\}$

$f'(x) = \dfrac{e^x(x+4) - e^x}{(x+4)^2} = \dfrac{e^x(x+3)}{(x+4)^2}$ $\qquad D_{f'} = \mathbb{R} \setminus \{-4\}$

$e^x(x+3) = 0 \quad / : e^x, \; e^x > 0$

$x + 3 = 0$

$x = -3$

$f\uparrow$: $f'(x) > 0$

$\dfrac{e^x(x+3)}{(x+4)^2} > 0 \quad \backslash \cdot (x+4)^2, \; (x+4)^2 > 0$

$e^x(x+3) > 0 \quad \backslash : e^x$

$x + 3 > 0$

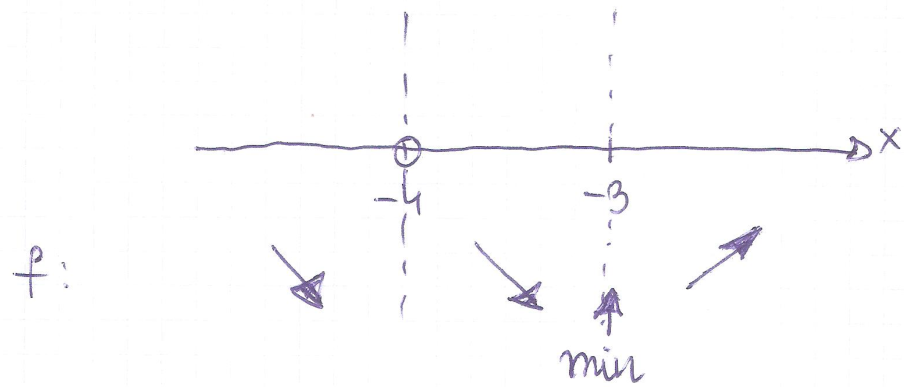
$x > -3$

$f\downarrow$: $f'(x) < 0$

$\dfrac{e^x(x+3)}{(x+4)^2} < 0$

$\vdots$

$x < -3$



$f_{min}(-3) = \dfrac{e^{-3}}{1} = \dfrac{1}{e^3}$

g) $f(x) = \frac{\ln x}{x}$

D_f: $x \neq 0$ i $x > 0$

$D_f = (0, +\infty)$

$f'(x) = \frac{\frac{1}{x} \cdot x - \ln x \cdot 1}{x^2}$

$f'(x) = \frac{1 - \ln x}{x^2}$

$D_{f'} = (0, +\infty)$

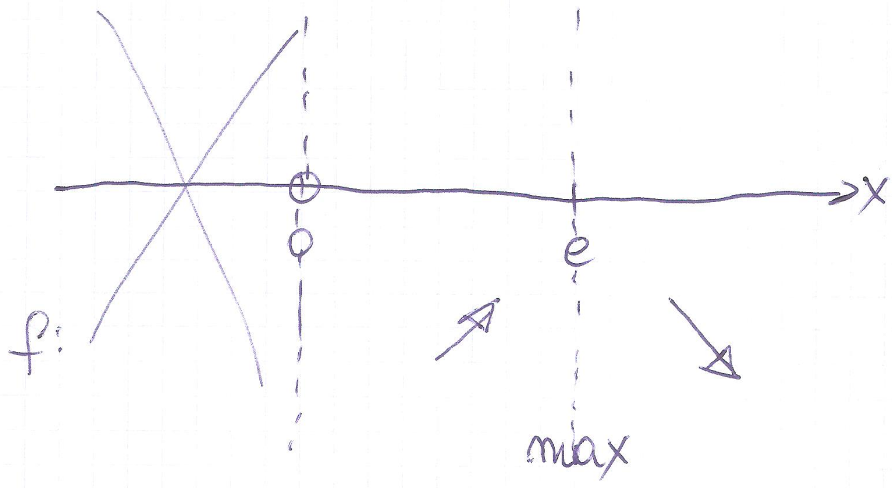
$f'(x) = 0 \Leftrightarrow \frac{1-\ln x}{x^2} = 0 \Rightarrow 1 - \ln x = 0$

$\ln x = 1$

$\ln x = \ln e$

$x = e$

$f \nearrow$: $f'(x) > 0$	$f \searrow$: $f'(x) < 0$
$\frac{1-\ln x}{x^2} > 0$	$\frac{1-\ln x}{x^2} < 0$
$1 - \ln x > 0$	$1 - \ln x < 0$
$1 > \ln x$	$1 < \ln x$
$\ln e > \ln x$	$\ln e < \ln x$
$e > x$	$e < x$



$f_{max}(e) = \frac{\ln e}{e} = \frac{1}{e}$

Z: $x > 0$, $a > 0$, $a \neq 1$, $\alpha \in \mathbb{R}$, $y > 0$

$\log_a x = b \Leftrightarrow a^b = x$

$1 = \log_a a$

$\alpha = \log_a a^\alpha$

$\log_a x = \log_a y \Leftrightarrow x = y$

$\log_a x < \log_a y$

$x < y$ dla $a > 1$

$x > y$ dla $a \in (0,1)$

$0 = \log_a 1$

$\alpha \log_a x = \log_a x^\alpha$

h) $f(x)=\frac{x}{\ln x}$

D_f: $x>0$ i $\ln x \neq 0$

$\ln x \neq \ln 1$

$x \neq 1$

$D_f = (0,1) \cup (1,+\infty)$

$D_{f'} = (0,1) \cup (1,+\infty)$

$$f'(x)=\frac{1\cdot \ln x - x\cdot \frac{1}{x}}{(\ln x)^2}$$

$$f'(x)=\frac{\ln x - 1}{(\ln x)^2}$$

$$\frac{\ln x-1}{(\ln x)^2}=0$$

$$\ln x - 1 = 0$$

$$\ln x = 1$$

$$x=e$$

$f\nearrow$: $f'(x)>0$

$$\frac{\ln x-1}{(\ln x)^2}>0$$

$$\ln x-1>0$$

$$\ln x>1$$

$$\ln x>\ln e$$

$$x>e$$

$f\searrow$: $f'(x)<0$

$$\frac{\ln x-1}{(\ln x)^2}<0$$

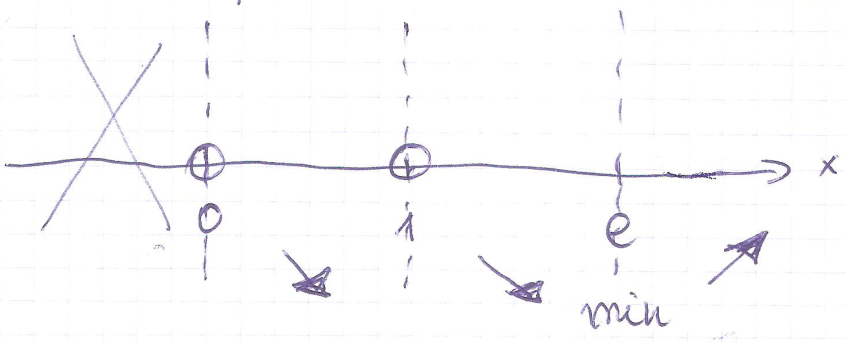
$$\ln x-1<0$$

$$\ln x<1$$

$$\ln x<\ln e$$

$$x<e$$

f:



$$f_{min}(e)=\frac{e}{\ln e}=e$$

i) $f(x) = x^2 \ln x$

$D_f: \ x>0$

$D_f = (0, +\infty)$

$D_{f'} = D_f$

$f'(x) = 2x\cdot \ln x + x^2 \cdot \frac{1}{x}$

$f'(x) = 2x\ln x + x = x(2\ln x + 1)$

$f'(x) = 0$

$x(2\ln x + 1) = 0$

$x = 0 \quad \vee \quad 2\ln x + 1 = 0$

$\uparrow$ sprzeczność

$2\ln x = -1$

$\ln x = \frac{-1}{2}$

$x = e^{-\frac{1}{2}}$

$f\uparrow: \quad f'(x) > 0$

$x(2\ln x + 1) > 0 \quad | : x \ (x>0)$

$2\ln x + 1 > 0$

$\ln x > -\frac{1}{2}$

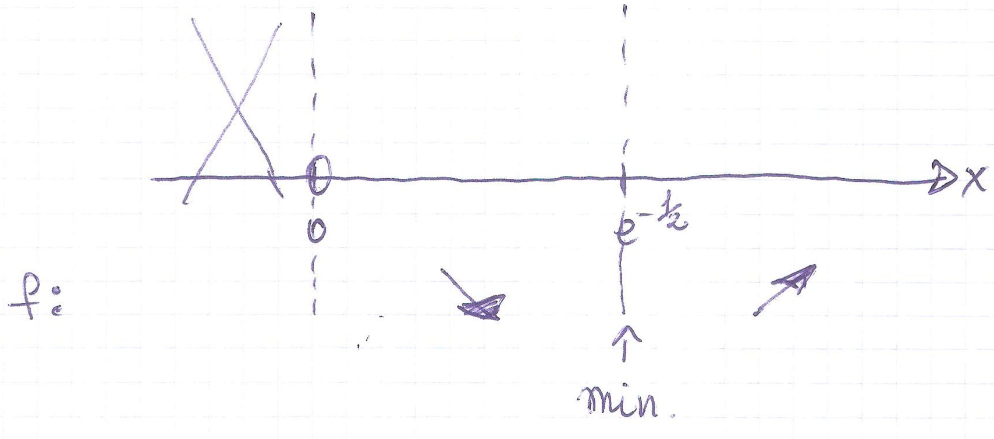
$\ln x > \ln e^{-\frac{1}{2}}$

$x > e^{-\frac{1}{2}}$

$f\downarrow: \quad f'(x) < 0$

$\vdots$

$x < e^{-\frac{1}{2}}$



$f_{min}(e^{-\frac{1}{2}}) = \ln(e^{-\frac{1}{2}})\cdot(e^{-\frac{1}{2}})^2 = -\frac{1}{2}\cdot e^{-1} = \frac{-1}{2e}$

j) $f(x) = e^{-x} \cdot x^3$ $\qquad D_f = \mathbb{R}$

$f'(x) = e^{-x} \cdot (-1) \cdot x^3 + e^{-x} \cdot 3x^2$ $\qquad D_{f'} = \mathbb{R}$

$f'(x) = -x^3 e^{-x} + 3x^2 e^{-x}$

$f'(x) = x^2 e^{-x}(3-x)$

$f'(x) = 0$

$x^2 e^{-x}(3-x) = 0 \quad | : e^{-x}, \; e^{-x} > 0$

$x^2(3-x) = 0$

$x = 0 \lor x = 3$

$f \nearrow$: $f'(x) > 0$

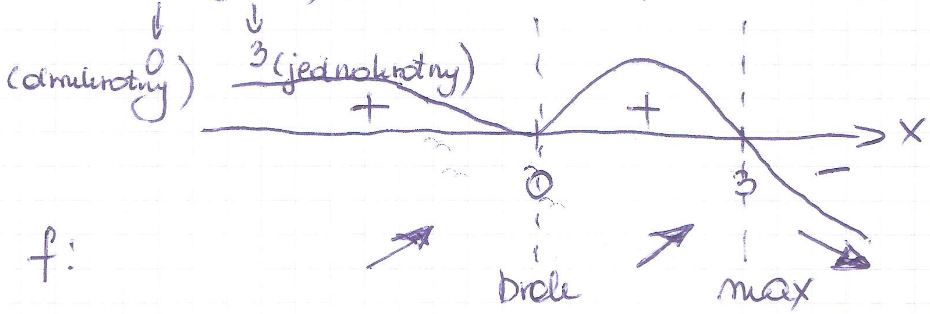
$x^2 e^{-x}(3-x) > 0 \quad | : e^{-x}$

$x^2(3-x) > 0$

$f \searrow$: $f'(x) < 0$

$\vdots$

$x^2(3-x) < 0$



$f_{max}(3) = e^{-3} \cdot 3^3 = \frac{27}{e^3}$

k) $f(x) = \frac{x^2}{2}\operatorname{arctg}(x+1) - \frac{x}{2} + \frac{1}{2}\ln(x^2+2x+2)$

$D_f = \mathbb{R}$

$f'(x) = x\cdot\operatorname{arctg}(x+1) + \frac{x^2}{2}\cdot\frac{1}{1+(x+1)^2} - \frac{1}{2} +$

$+ \frac{1}{2}\cdot\frac{1}{x^2+2x+2}\cdot(2x+2)$

$f'(x) = x\operatorname{arctg}(x+1) + \frac{1}{2}\cdot\frac{x^2}{x^2+2x+2} - \frac{1}{2} + \frac{1}{2}\cdot\frac{2x+2}{x^2+2x+2}$

$f'(x) = x\operatorname{arctg}(x+1) + \frac{1}{2}\cdot\frac{x^2}{x^2+2x+2} - \frac{1}{2}\cdot\frac{x^2+2x+2}{x^2+2x+2} + \frac{1}{2}\cdot\frac{2x+2}{x^2+2x+2}$

$f'(x) = x\operatorname{arctg}(x+1)$

$f'(x) = 0 \iff x\operatorname{arctg}(x+1) = 0$

$x = 0$ lub $\operatorname{arctg}(x+1) = 0$

$x+1 = 0$

$x = -1$

$f\nearrow$: $f'(x) > 0$

$x\operatorname{arctg}(x+1) > 0$

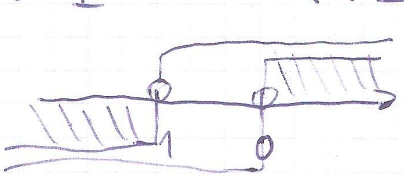
$x > 0$ i $\operatorname{arctg}(x+1) > 0$

lub

$x < 0$ i $\operatorname{arctg}(x+1) < 0$

$\begin{cases} x > 0 \\ (x+1) > 0 \end{cases}$ lub $\begin{cases} x < 0 \\ x+1 < 0 \end{cases}$

$\begin{cases} x > 0 \\ x > -1 \end{cases}$ lub $\begin{cases} x < 0 \\ x < -1 \end{cases}$



$x \in (-\infty, -1) \cup (0, +\infty)$

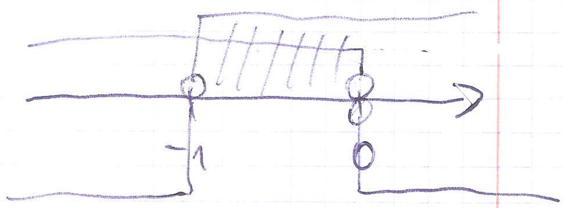
$f\searrow$: $f'(x) < 0$

$x\operatorname{arctg}(x+1) < 0$

$\begin{cases} x < 0 \\ \operatorname{arctg}(x+1) > 0 \end{cases}$ lub $\begin{cases} x > 0 \\ \operatorname{arctg}(x+1) < 0 \end{cases}$

$\begin{cases} x < 0 \\ x+1 > 0 \end{cases}$ lub $\begin{cases} x > 0 \\ x+1 < 0 \end{cases}$

$\begin{cases} x < 0 \\ x > -1 \end{cases}$ lub $\begin{cases} x > 0 \\ x < -1 \end{cases}$



$x \in (-1, 0)$

$f_{max}(-1) = \frac{1}{2}$

$f_{min}(0) = \frac{\ln 2}{2}$

f: $\nearrow$ -1 max $\searrow$ 0 min $\nearrow$

Zadanie 3 Wyznaczyć punkty przegięcia oraz przedziały wypukłości i wklęsłości funkcji:

$$f(x) = \frac{1}{56}x^8 - \frac{1}{42}x^7 - \frac{1}{30}x^6 + \frac{1}{20}x^5$$

$D_f = \mathbb{R}$

$$f'(x) = \frac{8}{59}x^7 - \frac{7}{42}x^6 - \frac{6}{30}x^5 + \frac{5}{20}x^4$$

$$f'(x) = \frac{1}{7}x^7 - \frac{1}{6}x^6 - \frac{1}{5}x^5 + \frac{1}{4}x^4 \quad , D_{f'} = \mathbb{R}$$

$$f''(x) = x^6 - x^5 - x^4 + x^3 \quad , D_{f''} = \mathbb{R}$$

$f''(x) = 0$

$x^6 - x^5 - x^4 + x^3 = 0$

$x^3(x^3 - x^2 - x + 1) = 0$

$x^3(x^2(x-1) - (x-1)) = 0$

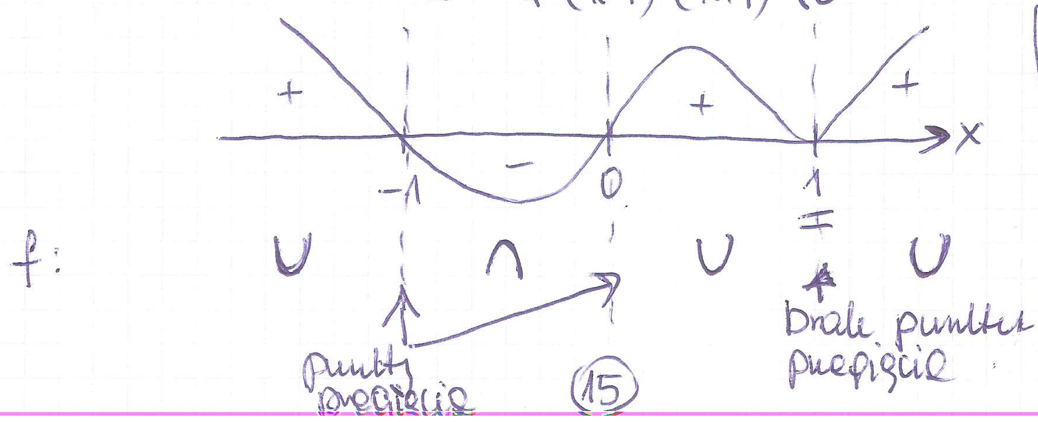
$x^3(x-1)(x^2-1) = 0$

$x^3(x-1)(x-1)(x+1) = 0$

$x^3(x-1)^2(x+1) = 0$

$x = 0$ (potrójny) lub $x = 1$ (podwójny) lub $x = -1$ (pojedynczy) (punkty w których może być punkt przegięcia)

- f jest wypukła gdy $f''(x) > 0$ $\quad (\cup)$
  $x^3(x-1)^2(x+1) > 0$
- f jest wklęsła gdy $f'(x) < 0$ $\quad (\cap)$
  $x^3(x-1)^2(x+1) < 0$



Punkty przegięcia:
$x = -1$, $x = 0$.

Zadanie 4 Wyznaczyć wartość największą i najmniejszą funkcji $f(x)=3x^4+8x^3-6x^2-24x$ w przedziale $x\in\langle -2,0\rangle$.

$f'(x)=12x^3+24x^2-12x-24$

$f'(x)=12x^2(x+2)-12(x+2)$

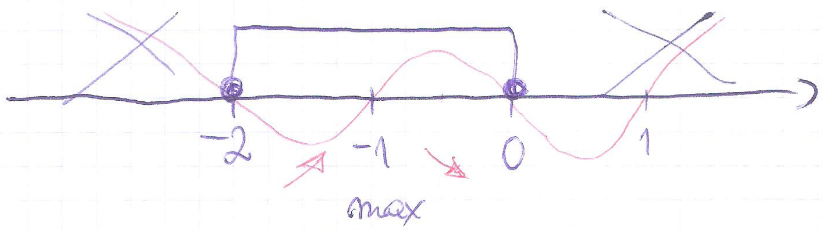
$f'(x)=(x+2)(12x^2-12)$

$f'(x)=12(x+2)(x^2-1)$

$f'(x)=12(x+2)(x-1)(x+1) \Rightarrow x=-2,\ x=-1,\ x=1 \notin\langle -2,0\rangle$

$f'(x)>0$ | $f'(x)<0$

$12(x+2)(x-1)(x+1)>0$ | $12(x+2)(x-1)(x+1)<0$



$f(-2)=8$

$f(-1)=13$ ← wartość największa

$f(0)=0$ ← wartość najmniejsza